Clarification: Here we clarify some notations and make a few corrections to maximize the consistence of our discussion.

Let X be a nonempty set.

- For every $A \subseteq 2^X$, the upwards closure $\uparrow A$ is defined by

  $$\uparrow A =_{def} \left\{ S \subseteq X \;\middle|\; B \subseteq S \text{ for some } B \in A \right\}.$$

  We abuse the notation by writing $\uparrow \{x\}$ for $\uparrow \{\{x\}\}$. ■

- For every $x \in X$, let $\mathcal{F}_x$ be a collection of filters such that,

  1. $\uparrow \{x\} \in \mathcal{F}_x$,
  2. for every $F \in \mathcal{F}_x$, $F \subseteq 2^X$ and F is a filter, and
  3. if $F \in \mathcal{F}_x$ and F' is a filter with $F \subseteq F'$, then $F' \in \mathcal{F}$. ■

  Note that, given an $x \in X$, $\mathcal{F}_x$ is not unique.

- We say that $(X, \mathcal{F})$ is a convergence space if and only if, for every $x \in X$, there is an $\mathcal{F}_x \in \mathcal{F}$. ■

- We say that $(X, \mathcal{T})$ is a topological space if and only if $\mathcal{T} \subseteq 2^X$ is a collection of open sets such that,

  1. $\emptyset \in \mathcal{T}$.
  2. $X \in \mathcal{T}$.
  3. $\mathcal{T}$ is closed under finite intersections.
  4. For every $\mathcal{S} \subseteq \mathcal{T}$, we have $\bigcup \mathcal{S} \in \mathcal{T}$. ■

Induced Topologies: (Exercise 6) Given any convergence space $(X, \mathcal{F})$, we can obtain an *induced topology* as follows.

For each $x \in X$, fix an $\mathcal{F}_x \in \mathcal{F}$ and define $\mathcal{N}_x = \bigcap \mathcal{F}_x$. By taking $\mathcal{N}_x$ as the set of neighborhoods of x for every $x \in X$, we can obtains a collection of open sets. Let this collection be $\mathcal{T}$. By exercise 4 of note III, $(X, \mathcal{T})$ is a topological space.

A Convergence Space Determined by a Topology: (Exercise 7) Given any topology space $(X, \mathcal{T})$, we can obtain a *convergence space* as follows.

For each $x \in X$, let $\mathcal{N}_x$ be the set of neighborhoods of x in topology $(X, \mathcal{T})$. For every $x \in X$, let $\mathcal{F}_x$ be the collection of all super-filters of $\mathcal{N}_x$. Also, let $\mathcal{F} = \{\mathcal{F}_x \mid x \in X\}$.

To see that $(X, \mathcal{F})$ is a convergence space, we shall prove that $\mathcal{N}_x$ is a filter for every $x \in X$. In other words, we need to prove that $\mathcal{N}_x$ is closed under finite intersection and superset.

1. Let $A, B \in \mathcal{N}_x$. By the definition of neighborhood, there are two open sets O_A and O_B such that, $x \in O_A$, $O_A \subseteq A$, $x \in O_B$, and $O_B \subseteq B$. Since O_A and O_B are two open sets, so is $O_A \cap O_B$. Moreover, $x \in O_A \cap O_B$. Since $(O_A \cap O_B) \subseteq (A \cap B)$, it follows that $A \cap B$ is a neighborhood of x and hence $A \cap B \in \mathcal{N}_x$.

2. Since any superset of a neighborhood of x is also a neighborhood of x, it follows that $\mathcal{N}_x$ is closed under superset.

Since $\uparrow \{x\}$ is a super-filter of $\mathcal{N}_x$, it follow that $\uparrow \{x\} \in \mathcal{F}_x$. Thus, $(X, \mathcal{F})$ is a convergence space. □

Definition 1 (Principal) *A set S is said to be principal if and only if $\bigcap S \in S$.* ■

According to the definition above, we say that $\mathcal{F}_x$ is principal iff $\bigcap \mathcal{F}_x \in \mathcal{F}_x$. Likewise, we say that $\bigcap \mathcal{F}_x$ is principal iff $\bigcap(\bigcap \mathcal{F}_x) \in \bigcap \mathcal{F}_x$.

Definition 2 (Pointed Convergence Spaces) *We say that a convergence space $(X, \mathcal{F})$ is pointed if and only if, for every $x \in X$, both $\mathcal{F}_x$ and $\bigcap \mathcal{F}_x$ are pointed.* ■

Reflexive Directed Graphs and Pointed Convergence Spaces

- Every pointed convergence space $(X, \mathcal{F})$ determines a reflexive directed graph (V, E), where $V = X$ and E is defined as follows. For convenience, let $\langle p, q \rangle$ denote an edge from node p to node q. E is the set of edges defined by:

$$\langle p, q \rangle \in E \text{ if and only if } p, q \in V \text{ and } q \in \bigcap(\bigcap \mathcal{F}_p).$$

  The only thing we need to prove is the reflexiveness of (V, E). Namely, for every $x \in X$, we have to show that $x \in \bigcap(\bigcap \mathcal{F}_x)$. By the definition of convergence space, $\uparrow \{x\} \in \mathcal{F}_x$. Thus, for every $A \in \bigcap \mathcal{F}_x$, we know that $A \in \uparrow \{x\}$, and hence $x \in A$. In other words, x is a member of every member of $\bigcap \mathcal{F}_x$, i,e., $x \in \bigcap(\bigcap \mathcal{F}_x)$. Therefore, $\langle x, x \rangle \in E$. □

  Note that, in the proof above we do not need the assumption that the convergence space is pointed.

- Every reflexive directed graph (V, E) determines a pointed convergence space as follows. For every $p \in V$, the principal neighborhood of p (denoted as $\mathcal{N}(p)$) is defined by
$$\mathcal{N}(p) =_{def} \{q \mid \langle p, q\rangle \in E\}.$$
Since (V, E) is reflexive, $p \in \mathcal{N}(p)$ for every $p \in V$. One can prove that $\uparrow \{\mathcal{N}(x)\}$ is a filter.[1] Let $\mathcal{F}_x$ be the collection of all super-filters of $\uparrow \{\mathcal{N}(x)\}$. Let $\mathcal{F} = \{\mathcal{F}_x \mid x \in X\}$. Since
$$A \in \uparrow \{\mathcal{N}(x)\} \Longrightarrow A \in \uparrow \{x\},$$
it follows that $\uparrow \{x\}$ is a super-filters of $\uparrow \{\mathcal{N}(x)\}$. Therefore $\uparrow \{x\} \in \mathcal{F}_x$, and hence $(X, \mathcal{F})$ is a convergence space.

  By the construction, we know that
$$\bigcap \mathcal{F}_x = \uparrow \{\mathcal{N}(x)\} \text{ and } \uparrow \{\mathcal{N}(x)\} \in \mathcal{F}_x.$$
Thus, $\mathcal{F}_x$ is principal. Also,
$$\bigcap(\bigcap \mathcal{F}_x) = \bigcap \uparrow \{\mathcal{N}(x)\} = \mathcal{N}(x) \text{ and } \mathcal{N}(x) \in \uparrow \{\mathcal{N}(x)\},$$
where $\uparrow \{\mathcal{N}(x)\} = \bigcap \mathcal{F}_x$. Thus, $\bigcap \mathcal{F}_x$ is also principal. Therefore, $(X, \mathcal{F})$ is pointed. □

Recall from Note II the definition of continuity on convergence spaces:

Definition 3 (Continuity) *Let $(X, \mathcal{F})$ and $(Y, \mathcal{D})$ be two convergence spaces and $f : X \to Y$ be a function. We say that f is continuous at $x \in X$ if and only if*
$$\forall\, F \in \mathcal{F}_x \left[\uparrow \hat{\hat{f}}(F) \in \mathcal{D}_{f(x)}\right]. \quad (1)$$
■

Our next goal is to see that, is there another way to formulate the continuity if the two convergence spaces $(X, \mathcal{F})$ and $(Y, \mathcal{D})$ are both pointed.

[1]Try to prove this statement.