**Corrections:**

- In the first lecture note, Definition I.2 for a convergence space should be that $\mathcal{F}$ is closed under superfilter. Thus,

  **Definition I:** $(X, \mathcal{F})$ is a *convergence space* iff $X$ is a set and $\mathcal{F}$ is a collection of *filters* such that:

  1. For every $F \in \mathcal{F}$, $F \subseteq 2^X$.
  2. If $F \in \mathcal{F}$ and $F'$ is a filter such that $F \subseteq F'$, then $F' \in \mathcal{F}$.
  3. For every $x \in X$, the set of all supersets of $\{x\}$ is a member of $\mathcal{F}$. ■

- In Lecture Note II the 4th line of the proof of Proposition 3, $y = B$ should be $y \in B$. That is,

$$\iff \; y \in B \qquad \text{because } B \subseteq \hat{f}(X), (\Leftarrow) \text{ holds.}$$

**Solutions to Exercises:** In this note, we give solutions to the 8 exercises discussed in the class. We first recall the definition of $\mathcal{F}_x$ which is a collection of filters in $X$ such that,

$(i)$ $\mathcal{F}_x$ is closed under superfilter. That is

$$\text{if } F \in \mathcal{F} \text{ and } F' \text{ is a filter with } F \subseteq F', \text{ then } F' \in \mathcal{F}.$$

$(ii)$ $\uparrow \{x\} \in \mathcal{F}_x$.

**Ex 1** *For every* $x \in X$, $X \in \bigcap \mathcal{F}_x$.

**Proof:** Let $x \in X$. Since each $F \in \mathcal{F}_x$ is a filter which is closed under superset, it follows that $X \in F$. Therefore, $X$ is a member of every filter in $\mathcal{F}_x$, and hence $X \in \bigcap \mathcal{F}_x$. □

**Ex 2** *For every* $x \in X$, $\bigcap \mathcal{F}_x$ *is a filter.*

**Proof:** Following the definition of a filter, we need to prove that $\bigcap \mathcal{F}_x$ is not empty and is closed under $(i)$ finite intersection and $(ii)$ superset.

By Ex 1, $X \in \bigcap \mathcal{F}_x$. Thus, $\bigcap \mathcal{F}_x$ is not empty.

- For $(i)$, let $A, B \in \bigcap \mathcal{F}_x$. Fix any $F \in \mathcal{F}_x$. It must be the case that $A \in F$ and $B \in F$. Since $F$ is a filter, it follows that $A \cap B \in F$. Therefore, $A \cap B$ is a member of every $F \in \mathcal{F}_x$. Thus, $A \cap B \in \bigcap \mathcal{F}_x$.
- For $(ii)$, let $A \in \bigcap \mathcal{F}_x$. Thus, $A \in F$ for every $F \in \mathcal{F}_x$. Let $A \subseteq B \subseteq X$. Since $F$ is a filter, $B \in F$. Therefore, $B$ is a member of every $F \in \mathcal{F}_x$. Thus, $B \in \bigcap \mathcal{F}_x$. □

**Ex 3** *For every $x \in X$, $\bigcap \mathcal{F}_x$ is the greatest lower bound of $\mathcal{F}_x$ with respect to $\subseteq$.*

**Proof:** We shall argue that $(i)$ $\bigcap \mathcal{F}_x$ is a lower bound of $\mathcal{F}_x$ and $(ii)$ if $A$ is a lower bound of $\mathcal{F}_x$ , then $A \subseteq \bigcap \mathcal{F}_x$.

- For $(i)$, fix any $F \in \mathcal{F}_x$. It is clear that $\bigcap \mathcal{F}_x \subseteq F$. Thus, $\bigcap \mathcal{F}_x$ is a lower bound of $\mathcal{F}_x$.
- For $(ii)$, suppose that $A$ is a lower bound of $\mathcal{F}_x$ and $a \in A$. Since $A$ is a lower bound of $\mathcal{F}_x$, $A \subseteq F$ for every $F \in \mathcal{F}_x$. Thus, if $a \in A$, then $a \in F$ for every $F \in \mathcal{F}_x$. Therefore, $a \in \bigcap \mathcal{F}_x$, and hence $A \subseteq \bigcap \mathcal{F}_x$. □

Here we formally define neighborhoods of an $x \in X$ and open sets in a topological space.

**Definition 1 (Neighborhoods and Open sets)** *Let $(X, \mathcal{F})$ be a convergence space and let $U \subseteq X$.*

- *Let $x \in U$. We say that $U$ is a* ***neighborhood*** *of $x$ iff $U \in \bigcap \mathcal{F}_x$.*
- *We say that $U$ is* ***open*** *iff $U$ is a neighborhood of every $x \in U$.* ■

Note: We now can consider $\bigcap \mathcal{F}_x$ as the collection of all neighborhoods of $x$.

**Ex 4** *The collection of all open subsets of $X$ forms a topology.*

**Proof:** Let $\mathcal{T}$ be the collection of all open subsets of $X$. Recall the definition of a topology. We prove the following four requirements for $\mathcal{T}$ being a topology in space $X$.

1. By definition $\emptyset$ is open. Thus, $\emptyset \in \mathcal{T}$.
2. From Ex 1, for every $x \in X$, $X \in \bigcap \mathcal{F}_x$. Thus, for every $x \in X$, $X$ is a neighborhood of $x$, and hence $X$ is open. Therefore, $X \in \mathcal{T}$.

3. Let $A, B \in \mathcal{T}$, i.e., $A$ and $B$ are both open. We want to prove that $A \cap B$ is also open. That is, we want to prove that, for every $p \in A \cap B$, $(A \cap B) \in \bigcap \mathcal{F}_p$. Fix any $p \in A \cap B$. Since $p \in A$ and $A$ is open, we have $A \in \bigcap \mathcal{F}_p$. Likewise, $p \in B$ and $B$ is open, we have $B \in \bigcap \mathcal{F}_p$. By Ex 2, $\bigcap \mathcal{F}_p$ is a filter; namely, $\bigcap \mathcal{F}_p$ is closed under finite intersection. Therefore, $(A \cap B) \in \bigcap \mathcal{F}_p$. Since for every $p \in A \cap B$, we have $(A \cap B) \in \bigcap \mathcal{F}_p$, by definition, $A \cap B$ is open.

4. Let $\{U_\lambda | \lambda \in \Lambda\}$ be a nonempty collection of open sets. We shall prove that $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open.[1] Fix any $p \in \bigcup_{\lambda \in \Lambda} U_\lambda$. Suppose that $p \in U_{\lambda'}$ for some $\lambda' \in \Lambda$. Since $U_{\lambda'}$ is open, it follows that $U_{\lambda'} \in \bigcap \mathcal{F}_p$. Since $\bigcap \mathcal{F}_p$ is a filter and $U_{\lambda'} \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$, we have $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{F}_p$. Therefore, $\bigcup_{\lambda \in \Lambda} U_\lambda$ is open. □

**Ex 5** *There is a topological space in which not every neighborhood of some points is open.*

**Sketch of Proof:** Consider $\mathbb{R}$ the set of real numbers. For every $a, b \in \mathbb{R}$, let $(a, b)$ denote an open set such that,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}. \tag{1}$$

We say that $A \subseteq \mathbb{R}$ is a *neighborhood* of $p \in \mathbb{R}$ if there is an open interval $(a, b) \subseteq A$ with $p \in (a, b)$. Also, let

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

Consider the example $[0, 1]$ and the point 0.5. Clearly, $[0, 1]$ is a neighborhood of 0.5 because 0.5 is a point in an open set, e.g., $(0.3, 0.7)$, and $(0.3, 0.7) \subseteq [0, 1]$. However, $[0, 1]$ is not a neighborhood of 0 and 1. This is because no open set that contains 0 or 1 can be a subset set of $[0, 1]$. Thus, $[0, 1]$ is not open in this topology defined by the open sets in form of (1).[2] □

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| Definition for Induced Topology and Ex 6 and Ex 7 will be prepared in Lecture Note IV |

**Ex 8** *Let $\Gamma : 2^X \to 2^X$ be defined with the following properties:*

*1.* $\Gamma(X) = X$.

[1] Note that, we require a topology to be closed under *arbitrary* union but not *finite* union. Thus, simply assume that $U$ and $V$ are open and prove that $U \cup V$ is open is not enough. On the other hand, we can prove that *arbitrary* intersection does not preserve open sets.

[2] In fact, open sets and neighborhoods are two equivalent approaches to define a topology. In our example, we define a topology space by giving the basic open sets in form of (1).

2. *$\Gamma(A) \subseteq A$, for all $A \subseteq X$.*

3. *$\Gamma(\Gamma(A)) = \Gamma(A)$, for all $A \subseteq X$.*

4. *$\Gamma(A \cap B) = \Gamma(A) \cap \Gamma(B)$, for all $A, B \subseteq X$.*

*Let $A \subseteq X$. We say that $A$ is a fixed points of $\Gamma$ iff $\Gamma(A) = A$. Prove that*

1. *$\emptyset$ is a fixed point of $\Gamma$.*

2. *If $U$ and $V$ are fixed points of $\Gamma$, then so is $U \cap V$.*

3. *If $\{U_\lambda | \lambda \in \Lambda\}$ is a collection of fixed points of $\Gamma$, then $\bigcup_{\lambda\in\Lambda} U_\lambda$ is also a fixed point of $\Gamma$.*

**Proof:**

1. By the second property of $\Gamma$, $\Gamma(\emptyset) \subseteq \emptyset$. Thus, $\Gamma(\emptyset) = \emptyset$

2. Let $\Gamma(U) = U$ and $\Gamma(V) = V$. We have

$$\Gamma(U \cap V) = G(U) \cap G(V) = U \cap V.$$

3. Let $\{U_\lambda | \lambda \in \Lambda\}$ be a collection of fixed points of $\Gamma$. We first prove that

$$A \subseteq B \Longrightarrow \Gamma(A) \subseteq \Gamma(B) \tag{2}$$

Let $A \subseteq B$. We have $A = A \cap B$. Thus,

$$\Gamma(A) = \Gamma(A \cap B) = \Gamma(A) \cap \Gamma(B) \subseteq \Gamma(B).$$

Fix any $\lambda' \in \Lambda$. We have $U_{\lambda'} \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$. By (2), we have

$$\begin{aligned} \Gamma(U_{\lambda'}) &\subseteq \Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda) \\ U_{\lambda'} &\subseteq \Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda) \qquad \text{because } U_{\lambda'} \text{ is a fixed point.} \end{aligned}$$

Since $\lambda'$ is any member in $\Lambda$, it follows that,

$$\bigcup_{\lambda\in\Lambda} U_\lambda \subseteq \Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda). \tag{3}$$

Also, by the second property of $\Gamma$,

$$\Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda. \tag{4}$$

(3) and (4) imply that,

$$\Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda) = \bigcup_{\lambda\in\Lambda} U_\lambda.$$

Therefore, $\bigcup_{\lambda\in\Lambda} U_\lambda$ is a fixed point of $\Gamma$. □

**Alternative proof:** Fix any $\lambda' \in \Lambda$. We have $U_{\lambda'} = U_{\lambda'} \cap (\bigcup_{\lambda\in\Lambda} U_\lambda)$. Thus,

$$\Gamma(U_{\lambda'}) = \Gamma(U_{\lambda'} \cap (\bigcup_{\lambda\in\Lambda} U_\lambda)) = \Gamma(U_{\lambda'}) \cap \Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda).$$

$$U_{\lambda'} = U_{\lambda'} \cap \Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda). \tag{5}$$

(5) implies that

$$U_{\lambda'} \subseteq \Gamma(\bigcup_{\lambda\in\Lambda} U_\lambda).$$

Then, we follow the same arguments in the previous proof to finish. □