

Notations and Terminologies: Let X, Y be sets, and $f : X \rightarrow Y$.

- Define $\hat{f} : 2^X \rightarrow 2^Y$ by

$$\hat{f}(A) = \{f(a) \mid a \in A\}.$$

- Define $\hat{f}^{-1} : 2^Y \rightarrow 2^X$ by

$$\hat{f}^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

- Define the range of function f by

$$\text{range}(f) = \hat{f}(X).$$

- Let X be the universe of the discourse. Suppose $\mathcal{F} \subseteq 2^X$. The upward closure of \mathcal{F} , denoted by $\uparrow \mathcal{F}$, is defined by

$$\uparrow \mathcal{F} = \{A \subseteq X \mid B \subseteq A \text{ for some } B \in \mathcal{F}\}.$$

Proposition 1 \hat{f} is monotone w.r.t. \subseteq .

$$\text{I.e., } \forall A, B \subseteq X \left[A \subseteq B \Rightarrow \hat{f}(A) \subseteq \hat{f}(B) \right].$$

Proposition 2 \hat{f}^{-1} is monotone w.r.t. \subseteq .

$$\text{I.e., } \forall A, B \subseteq Y \left[A \subseteq B \Rightarrow \hat{f}^{-1}(A) \subseteq \hat{f}^{-1}(B) \right].$$

Proposition 3 $\text{range}(\hat{f}) = 2^{\text{range}(f)}$.

Proof: Suppose that $B \subseteq \hat{f}(X)$. Then, we have

$$\begin{aligned} y \in \hat{f}(\hat{f}^{-1}(B)) &\iff \exists x \in \hat{f}^{-1}(B) [f(x) = y] \\ &\iff \exists x \in X [f(x) \in B \wedge f(x) = y] \\ &\iff y \in B \quad \text{because } B \subseteq \hat{f}(X), (\iff) \text{ holds.} \end{aligned}$$

Thus,

$$B \subseteq \hat{f}(X) \implies B = \hat{f}(\hat{f}^{-1}(B)).$$

Since

$$[B = \hat{f}(\hat{f}^{-1}(B))] \implies [B \in \text{range}(\hat{f})],$$

we have

$$B \in 2^{\text{range}(f)} \iff B \subseteq \text{range}(f) \iff B \subseteq \hat{f}(X) \implies B \in \text{range}(\hat{f}).$$

Therefore, $\text{range}(\hat{f}) \supseteq 2^{\text{range}(f)}$. For $\text{range}(\hat{f}) \subseteq 2^{\text{range}(f)}$, see the proof in the following alternative proof. \square

Alternative proof for Proposition 3:

$$\begin{aligned}
B \in \text{range}(\hat{f}) &\iff \exists A \in 2^X \left[\hat{f}(A) = B \right] && \text{def. of range} \\
&\iff \exists A \subseteq X \left[\hat{f}(A) = B \right] && \text{def. of powerset} \\
&\iff \forall y \in B \exists x \in X [f(x) = y] && \text{see } \star \\
&\iff \forall y \in B \left[y \in \hat{f}(X) \right] \\
&\iff \forall y \in B [y \in \text{range}(f)] \\
&\iff B \subseteq \text{range}(f) \\
&\iff B \in 2^{\text{range}(f)}.
\end{aligned}$$

★ We shall prove that

$$\begin{aligned}
\exists A \subseteq X \left[\hat{f}(A) = B \right] &\iff \forall y \in B \exists x \in X [f(x) = y] \\
\exists A \subseteq X \left[\hat{f}(A) = B \right] &\implies \exists A \subseteq X \forall y \in B \left[y \in \hat{f}(A) \right] \\
&\implies \exists A \subseteq X \forall y \in B \exists x \in A [f(x) = y] \\
&\implies \exists A \subseteq X \forall y \in B \exists x \in X [f(x) = y] \\
&\implies \forall y \in B \exists x \in X [f(x) = y].
\end{aligned}$$

The reason for the last step above is that: The quantified variable A does not occur in $\forall y \in B \exists x \in X [f(x) = y]$.

$$\begin{aligned}
\forall y \in B \exists x \in X [f(x) = y] &\implies \forall y \in B \exists x \in \hat{f}^{-1}(B) [f(x) = y] \\
&\implies \forall y \in B \left[y \in \hat{f}(\hat{f}^{-1}(B)) \right] \\
&\implies B \subseteq \hat{f}(\hat{f}^{-1}(B)) \\
&\implies B \subseteq \hat{f}(\hat{f}^{-1}(B)) \wedge \hat{f}(\hat{f}^{-1}(B)) \subseteq B \\
&\implies B = \hat{f}(\hat{f}^{-1}(B)) \\
&\implies \exists A \subseteq X \left[\hat{f}(A) = B \right].
\end{aligned}$$

Note that $\hat{f}(\hat{f}^{-1}(B)) \subseteq B$ is a theorem. □

Proposition 4 $B = \hat{f}(\hat{f}^{-1}(B))$ iff $B \subseteq \text{range}(f)$.

Proof:

$$\begin{aligned}
B = \hat{f}(\hat{f}^{-1}(B)) &\iff B \subseteq \hat{f}(\hat{f}^{-1}(B)) \wedge \hat{f}(\hat{f}^{-1}(B)) \subseteq B \\
&\iff \forall y \in B \left[y \in \hat{f}(\hat{f}^{-1}(B)) \right] \\
&\iff \forall y \in B \exists x \in \hat{f}^{-1}(B) [f(x) = y] \\
&\iff \forall y \in B \exists x \in X [f(x) = y] \\
&\iff B \subseteq \text{range}(f).
\end{aligned}$$

□

Proposition 5 $\hat{f}(A \cap B) \subseteq \hat{f}(A) \cap \hat{f}(B)$.

Proof:

$$\begin{aligned}
 y \in \hat{f}(A \cap B) &\iff \exists x \in (A \cap B) [f(x) = y] \\
 &\iff \exists x [x \in A \wedge x \in B \wedge f(x) = y] \\
 &\iff \exists x [(x \in A \wedge f(x) = y) \wedge (x \in B \wedge f(x) = y)] \\
 &\implies \exists x [x \in A \wedge f(x) = y] \wedge \exists x [x \in B \wedge f(x) = y] \\
 &\iff y \in \hat{f}(A) \wedge y \in \hat{f}(B) \\
 &\iff y \in \hat{f}(A) \cap \hat{f}(B)
 \end{aligned}$$

□

Proposition 6 $A \cap B \subseteq \hat{f}^{-1}(\hat{f}(A) \cap \hat{f}(B))$.

Proof:

$$\begin{aligned}
 x \in A \cap B &\implies f(x) \in \hat{f}(A \cap B) \\
 &\iff x \in \hat{f}^{-1}(\hat{f}(A \cap B)) \\
 &\implies x \in \hat{f}^{-1}(\hat{f}(A) \cap \hat{f}(B)) \quad \text{by Propositions 5 and 2.}
 \end{aligned}$$

□

Proposition 7 Let $f : X \rightarrow Y$ and $\mathcal{F} \subseteq 2^X$ be a filter in space X . Then, $\hat{f}(\mathcal{F})$ is closed under finite intersection.

Proof: Let $U, V \in \hat{f}(\mathcal{F})$. Then, there exist $A, B \in \mathcal{F}$ such that,

$$\hat{f}(A) = U \quad \text{and} \quad \hat{f}(B) = V.$$

Since \mathcal{F} is a filter, \mathcal{F} is closed under finite intersection and hence $A \cap B \in \mathcal{F}$. By Proposition 6 we have

$$A \cap B \subseteq \hat{f}^{-1}(\hat{f}(A) \cap \hat{f}(B)) = \hat{f}^{-1}(U \cap V).$$

Since \mathcal{F} is a filter, \mathcal{F} is closed under superset and hence $\hat{f}^{-1}(U \cap V) \in \mathcal{F}$. Since

$$U \cap V = \hat{f}(A) \cap \hat{f}(B) \subseteq \hat{f}(X),$$

by Proposition 4, we have

$$\hat{f}(\hat{f}^{-1}(U \cap V)) = U \cap V.$$

Therefore, $U \cap V \in \hat{f}(\mathcal{F})$.

□

Example 1 *There is a function $f : X \rightarrow Y$ such that, for some filter \mathcal{F} in X , $\hat{f}(\mathcal{F})$ is not a filter in Y .*

By Proposition 7, we know that $\hat{f}(\mathcal{F})$ is closed under finite intersection. Thus, we have to show that $\hat{f}(\mathcal{F})$ is not closed under superset in space Y .

Let $X = Y = \{0, 1\}$. Define f by $f(0) = f(1) = 0$. Let filter $\mathcal{F} = 2^X$. We have

$$\begin{aligned}\hat{f}(\mathcal{F}) &= \{\hat{f}(A) \mid A \in \mathcal{F}\} \\ &= \{\hat{f}(A) \mid A \in 2^X\} \\ &= \{\hat{f}(A) \mid A \subseteq X\} \\ &= \{\emptyset, \{0\}\}.\end{aligned}$$

Clearly, $\{\emptyset, \{0\}\}$ is not closed under superset in space Y . Thus, $\hat{f}(\mathcal{F})$ is not a filter w.r.t. Y . \square

Proposition 8 *Let X be the universe of the discourse. Suppose that $\mathcal{F} \subseteq 2^X$ is closed under finite intersection. Then, $\uparrow \mathcal{F}$ is a filter.*

Proof: Let $A, B \in \uparrow \mathcal{F}$. By the definition of $\uparrow \mathcal{F}$, there are $C, D \in \mathcal{F}$ such that,

$$C \subseteq A \text{ and } D \subseteq B.$$

For such C and D , by the assumption, $C \cap D \in \mathcal{F}$. Since

$$(C \cap D) \subseteq (A \cap B),$$

it follows that

$$A \cap B \in \uparrow \mathcal{F}.$$

Thus, $\uparrow \mathcal{F}$ is closed under finite intersection. By the definition of $\uparrow \mathcal{F}$, it is clear that $\uparrow \mathcal{F}$ is closed under superset. Therefore, $\uparrow \mathcal{F}$ is a filter. \square

Proposition 9 *Let X be the universe of the discourse and $\mathcal{D} \subseteq 2^X$. Then*

$$\uparrow \uparrow \mathcal{D} = \uparrow \mathcal{D}.$$

Definition 1 *Let $\mathcal{D} \subseteq 2^X$. We say that \mathcal{D} is a **filterbase** iff*

$$\forall A, B \in \mathcal{D} \exists C \in \mathcal{D} [C \subseteq A \cap B].$$

Comment: The definition of filterbase for \mathcal{D} does not imply that \mathcal{D} is closed under finite intersection. For example, $\{\emptyset, \{a, b\}, \{a, c\}\}$ is a filterbase but not closed under finite intersection.

Proposition 10 *Let X be the universe of the discourse. Suppose that $\mathcal{D} \subseteq 2^X$ is a filterbase. Then, $\uparrow \mathcal{D}$ is a filter.*

Proof: Since \mathcal{D} is a filterbase, it follows that $\uparrow \mathcal{D}$ is closed under finite intersection.

- By Proposition 8, we have that $\uparrow \uparrow \mathcal{D}$ is a filter.
- By Proposition 9, $\uparrow \uparrow \mathcal{D} = \uparrow \mathcal{D}$ and hence $\uparrow \mathcal{D}$ is a filter. □

Notations: For each $x \in X$, let \mathcal{F}_x denote the collection of filters in X such that, (i) \mathcal{F}_x is closed under superfilter, i.e., if $F_1 \in \mathcal{F}$ is a filter and $F_1 \subseteq F_2$ is also a filter, then $F_2 \in \mathcal{F}$, and (ii) $\uparrow \{x\} \in \mathcal{F}_x$. The same notation applies to \mathcal{D}_y .

Definition 2 (Continuity) *Let $f : X \rightarrow Y$ where (X, \mathcal{F}) and (Y, \mathcal{D}) are convergence spaces. We say that f is continuous at x iff*

$$\forall F \in \mathcal{F}_x \left[\uparrow \hat{f}(F) \in \mathcal{D}_{f(x)} \right].$$