Question 1: Let X be set. Suppose that $F_1 \subseteq 2^X$ and $F_1 \subseteq 2^X$ are two filters. Prove that $F_1 \cap F_2$ is a filter.

Proof:

1. $F_1 \cap F_2$ is nonempty.

   Since F_1 and F_2 are two filters in X, thus $X \in F_1$ and $X \in F_2$. It follows that $X \in \mathcal{F}_1 \cap F_2$. Thus, $F_1 \cap F_2$ is nonempty.

2. $F_1 \cap F_2$ is closed under intersection.

   Suppose $A, B \in F_1 \cap F_2$. We shall prove that $A \cap B \in F_1 \cap F_2$.

   $$A \in (F_1 \cap F_2) \Rightarrow (A \in F_1 \ \wedge \ A \in F_2);$$
   $$B \in (F_1 \cap F_2) \Rightarrow (B \in F_1 \ \wedge \ B \in F_2).$$

   Since F_1 and F_2 are filters, we have

   $$(A \in F_1 \ \wedge \ B \in F_1) \Rightarrow (A \cap B) \in F_1;$$
   $$(A \in F_2 \ \wedge \ B \in F_2) \Rightarrow (A \cap B) \in F_2.$$

   Therefore,

   $$A \cap B \in F_1 \cap F_2.$$

3. $F_1 \cap F_2$ is closed under superset.

   Let $A \in (F_1 \cap F_2)$. Thus, $A \in F_1$ and $A \in F_2$. Let $A \subseteq B$ with $B \subseteq X$. Since F_1 and F_2 are both filters, it follows that $B \in F_1$ and $B \in F_2$. Therefore, $B \in (F_1 \cap F_2)$. □

Question 2: **Part a:** If F is a filter attached to p, then $F \subseteq [p]$.

Proof: Suppose that F is a filter attached to p. For any $A \in F$, by Definition 0.1, $p \in A$. By the definition of $[p]$, $A \in [p]$. Therefore, $F \subseteq [p]$. □

Part b: Prove that $[p]$ is the maximum filter attached to p.

Proof: For this problem, we have to prove that (i) $[p]$ is a filter attached to p and (ii) it is the maximum one. For (i), we prove the following:

1. Since $p \in X$, it follows that $X \in [p]$. Thus, $[p]$ is not empty.
2. Suppose $A, B \in [p]$. By definition, $p \in A$ and $p \in B$. Thus, $p \in A \cap B$ and $A \cap B \in [p]$. . Therefore, $[p]$ is closed under intersection.
3. Let $A \in [p]$. Suppose $B \subseteq X$ with $A \subseteq B$. Since $A \in [p]$, we have $p \in A$. Since $A \subseteq B$, we have $p \in B$, and thus $B \in [p]$. Therefore p is closed under superset. □

Therefore, $[p]$ is a filter and, by its definition, it is attached to p.

For (ii), suppose M is a maximal filter attached to p. Since $[p]$ is a filter attached to p and by the assumption on M, we have $[p] \subseteq M$. By **Part a**, $M \subseteq [p]$. Thus, $M = [p]$. Therefore, $[p]$ is the maximum filter attached to p. □

Question 3: Let $f : X \to Y$. Suppose that $G \subseteq 2^Y$ is a filter. Define F by

$$F = \left\{ \hat{f}^{-1}(V) \mid V \in G \right\}.$$

Prove that $\uparrow F$ is a filter.

Proof:

1. Since G is a filter, G is not empty. Thus, F is not empty, neither is $\uparrow F$.
2. Let $A'_X \in \uparrow F$ and $B'_X \in \uparrow F$. Thus, there are A_x and B_x such that,

   $$A_x \subseteq A'_x, A_x \in F \text{ and} B_x \subseteq B'_x, B_x \in F.$$

   Therefore, there are $A_Y, B_Y \in G$ such that,

   $$\hat{f}^{-1}(A_Y) = A_X \text{ and } \hat{f}^{-1}(B_Y) = B_X.$$

   Since G is a filter, $A_Y \cap B_Y \in G$. Thus, $\hat{f}^{-1}(A_Y \cap B_Y) \in F$. Since

   $$\hat{f}^{-1}(A_Y \cap B_Y) \subseteq \hat{f}^{-1}(A_Y) \cap \hat{f}^{-1}(B_Y),$$

   we have

   $$\hat{f}^{-1}(A_Y) \cap \hat{f}^{-1}(B_Y) \in \uparrow F.$$

   Thus, $A_X \cap B_X \in \uparrow F$. Moreover, since $A_x \subseteq A'_x$ and $B_x \subseteq A'_x$, it follows that $(A_X \cap B_Y) \subseteq (A'_X \cap B'_X)$. Thus,

   $$A'_X \cap B'_X \in \uparrow F.$$

3. By the definition of the upward closure, $\uparrow F$ is closed under superset.

   Therefore, $\uparrow F$ is a filter. □

Solution to Question 4:

input	output
$\emptyset$	$\emptyset$
$\{0\}$	$\{0\}$
$\{1\}$	$\{1, 6\}$
$\{2\}$	$\{3, 4\}$
$\{0, 1\}$	$\{0, 1, 6\}$
$\{0, 2\}$	$\{0, 3, 4\}$
$\{1, 2\}$	$\{1, 3, 4, 6\}$
$\{0, 1, 2\}$	$\{0, 1, 3, 4, 6\}$